Canonical Hamiltonian gyrocenter variables and gauge invariant representation of the gyrokinetic equation

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(Received 21 February 2001; revised manuscript received 26 July 2001; published 18 October 2001)

By using the extended phase space Hamiltonian Lie-transform perturbation method, Hamiltonian canonical variables have been found to describe the motion of gyrocenters. A representation of the gyrokinetic equation has been established in terms of the magnetic moment M, the total energy U, and the canonical toroidal momentum P of the particle. This representation of the gyrokinetic equation is invariant with respect to the gauge transformation of perturbation fields. It explicitly reveals the effects of toroidal symmetry breaking, and it indicates the role that the perturbed canonical toroidal momentum plays in the gyrokinetic theory. In particular, it is found that the free energy associated with $\partial_P f_0(M,U,P)$ [here $f_0(M,U,P)$ is the equilibrium distribution function] does not have any nonadiabatic linear driving to the axisymmetric modes.

DOI: 10.1103/PhysRevE.64.056404 PACS number(s): 52.25.Dg, 52.55.Fa, 52.35.Qz, 52.35.Bj

I. INTRODUCTION

One of the most challenging problems in tokamak controlled nuclear fusion research is to understand the anomalous transport of plasmas induced by microinstabilities or kinetic-magnetohydrodynamic (MHD) modes. To systematically investigate this problem, the gyrokinetic equation (GKE) [1–14] shall be used to describe the kinetic response of plasmas to the modes. Many papers have been published to discuss the derivation of the GKE [1–9]. Among these papers, the classical gyrokinetic theory [1–3] uses recursive method and involves explicitly gyroaveraging the Vlasov equation; the modern gyrokinetic theory [4–9] uses the Hamiltonian Lie-transform perturbation method [15–19]. GKE's in all of these papers are represented in terms of noncanonical variables.

When linearizing the GKE, one considers a small deviation of the system from the toroidal equilibrium. For an axisymmetric torus in equilibrium, it is well known that in addition to the magnetic moment M, there are two other invariants of motion: the total energy U and the canonical toroidal momentum P, which are associated with the temporal symmetry and the toroidal symmetry, respectively. Consequently, the equilibrium distribution function can be represented as a function of these three invariants of motion, $f_0(M,U,P)$, when the effects of finite banana width (FBW) are retained [20]. Therefore, in order to exploit this property of the equilibrium distribution function as much as possible, one is led to the consideration of canonical variables in formulating the GKE.

To investigate the effects of FBW on the kinetic-MHD instabilities, Porcelli, Stankiewicz, Kerner, and Berk (PSKB) have discussed the formulation of the drift kinetic equation (DKE) using (M,U,P) in dropping the effects of finite Larmor radius (FLR) [11]. PSKB's DKE has revealed the effects of toroidal symmetry breaking introduced by the perturbations; it indicates that the free energy associated with $\partial_P f_0(M,U,P)$ does not have any nonadiabatic linear driving to the axisymmetric modes. This is a very important issue in plasma physics.

For example, the $(m,n)=(1,0)$ (m and n are poloidal and toroidal mode numbers, respectively) and $(m,n)=(2,0)$ global shear Alfven eigenmodes discussed in Ref. [21] are clearly toroidal axisymmetric modes. Therefore, the excitation of these modes by energetic ions should be carefully reexamined. Another example is related to the suppression of turbulent transport by the sheared flows; it has been recently recognized that the axisymmetric modes are very important in evaluating the anomalous transport [12]. In understanding the behavior of the axisymmetric modes, it is crucially important that there is no nonadiabatic linear driving to the modes with $n=0$. We observed that this important statement was reached in Ref. [12] based on the classical GKE [3] formulated with the eikonal ansatz. Clearly, the conventional eikonal ansatz [1–3] breaks down at $n=0$. Essentially, a modified eikonal ansatz was used in Ref. [12] by assuming that the poloidal wavelength is much longer than the radial wavelength. However, it is the poloidal angle dependence of the perturbed fields that appears to linearly drive the axisymmetric modes in the classical gyrokinetic theory [1–3]; note that in the literature the diamagnetic drift frequency ω_* used in GKE is proportional to k_θ, the poloidal component of the wave vector [13].

To resolve the issues raised above, PSKB's DKE is not adequate since it does not include the effects of FLR. We observed that even for $n=0,1$, the perpendicular wavelength may still be comparable to the ion Larmor radius. It is unclear whether the effects of FLR shall change the important statement drawn from PSKB's DKE. Therefore, we have to write the GKE in the form similar to PSKB's DKE and make the GKE valid for arbitrary n. This has been partially solved in Refs. [10,22].

In Ref. [10], Gorelenkov, Cheng, and Fu (GCF) put the classical GKE [1] in the form similar to PSKB's DKE. GCF's GKE is written in eikonal form [10] and is based on the classical gyrokinetic theory [1]; it breaks down when $n=0,1$. Therefore, GCF's GKE has not answered the question

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we have. In Ref. [22], using axisymmetric cylinder geometry and assuming $f_0=f_0(U,P)$, Berk and Pfirsch have clearly shown that the free energy associated with $\partial_P f_0(U,P)$ does not have any linear driving to the axisymmetric modes, even with effects of FLR included. However, the general equilibrium distribution function in axisymmetric tokamaks should be $f_0(M,U,P)$. Therefore, the problem that we have is still there.

Another important advantage of PSKB's DKE is that it is invariant with respect to the gauge transformation of perturbation fields, although this was not explicitly indicated in their paper. To our knowledge, all of GKE's for electromagnetic perturbations do not have this advantage, although some authors claimed or implied that their GKE had this advantage, and this shall be made clear in the main text.

Since the modern gyrokinetic theory [4–9] is valid for modes with arbitrary *n*, we shall establish a GKE that has the advantage of the modern GKE and the advantages of PSKB's DKE.

In this paper, beginning with the Hamiltonian canonical variables [23,24] (in contrast to the previous modern gyrokinetic theory [4–9], where noncanonical Hamiltonian Lie transform was used), we shall use the canonical Hamiltonian Lie-transform perturbation method to systematically establish a representation of GKE in terms of (M,U,P). The representation shall explicitly reveal the effects of toroidal symmetry breaking and clearly show that there is no nonadiabatic linear driving to arbitrary axisymmetric modes, in contrast to the previous classical [1–3] and modern [4–9] gyrokinetic theory. And the new representation of GKE is invariant with respect to the gauge transformation of perturbation fields. The canonical Hamiltonian Lie-transform perturbation method used in this paper may also be of interest to readers working in general physics.

The remaining part of this paper is organized as follows. In Sec. II, we briefly review the theory of canonical Hamiltonian variables and the Lie-transform perturbation method. In Sec. III, we derive the canonical variables for gyrocenters (here and throughout this paper, following the terminology of Brizard [4], we denote the guiding centers in perturbed field as gyrocenters). In Sec. IV, we derive the new representation of GKE. In Sec. V, we present limiting cases and compare them with existing theory. In Sec. VI, we discuss the invariance with respect to gauge transformation of perturbation fields. In Sec. VII, we present the final gauge invariant form of GKE. In Sec. VIII, we summarize the main results and make some further discussions.

II. REVIEW OF CANONICAL HAMILTONIAN FORMULATION OF GUIDING-CENTER MOTION AND LIE-TRANSFORM PERTURBATION METHOD

In this section, we briefly review the well-established theory of canonical Hamiltonian formulation of the guiding-center motion [23,24] and the Lie-transform perturbation method [15–19], based on which our following discussions shall develop. For the details, we refer the readers to the literature.

A. Canonical Hamiltonian formulation of guiding-center motion

To begin our discussion, we write down the general representation of the equilibrium field in tokamaks.

$$\mathbf{B}=\nabla\psi_T\times\nabla\theta+\nabla\zeta\times\nabla\psi \tag{1a}$$

$$=g(\psi)\nabla\zeta+I(\psi)\nabla\theta+\delta(\psi,\theta)\nabla\psi, \tag{1b}$$

$$\Phi=\Phi(\psi,\theta), \tag{2}$$

where $\mathbf{B}$ is the equilibrium magnetic field and Φ is the electrostatic potential. (ψ,θ,ζ) are the magnetic flux coordinates; ψ is the poloidal magnetic flux, θ is the poloidal angle, and ζ is the toroidal angle. $\psi_T=\psi_T(\psi)$ is the toroidal magnetic flux, which satisfies $d\psi_T/d\psi=q(\psi)$, with $q(\psi)$ being the well-known MHD safety factor.

Guiding-center canonical variables are $(M,P,P_\theta;\xi,\zeta,\theta)$; M is the usual magnetic moment and ξ is the gyrophase angle. The canonical toroidal momentum P and the canonical poloidal momentum P_θ are defined by

$$P=g\rho_\parallel-\psi, \tag{3a}$$

$$P_\theta=I\rho_\parallel+\psi_T, \tag{3b}$$

where $\rho_\parallel=v_\parallel/B$, $v_\parallel$ is the parallel velocity. Note that with Eq. (3) we have

$$\psi=\psi(P,P_\theta), \tag{4a}$$

$$\rho_\parallel=\rho_\parallel(P,P_\theta). \tag{4b}$$

Guiding-center Hamiltonian is written as

$$H_0(M,P,P_\theta,\theta)=MB+\frac{1}{2}\rho_\parallel^2B^2+\Phi, \tag{5}$$

where, and throughout this paper, we have set $e_s=m_s=1$ (e_s and m_s are electric charge and mass of the particle, respectively) to make the formulas concise. The physical formulas can be obtained by restoring the factors of e_s and m_s.

In order to facilitate the following discussions on the perturbed time-dependent system, we introduce the extended phase space [16] $(M,P,P_\theta,-U;\xi,\zeta,\theta,t)$ and the independent parameter τ. This is a well-known tactic in dealing with the time-dependent system [16,7]. The fundamental one-form in this extended phase space can be written as

$$\Gamma_0=\Gamma_{0i}\,dz^i \tag{6a}$$

$$=M\,d\xi+P\,d\zeta+P_\theta\,d\theta-U\,dt-h_0\,d\tau, \tag{6b}$$

$$h_0=H_0-U, \tag{6c}$$

where we have used the notation $(z^1,z^2,z^3,z^4;z^5,z^6,z^7,z^8;z^9)\equiv(M,P,P_\theta,-U;\xi,\zeta,\theta,t;\tau)$. The equations of motion are readily found

$$\frac{d}{d\tau}\xi=\frac{\partial}{\partial M}h_0=\frac{\partial}{\partial M}H_0, \tag{7a}$$

$$\frac{d}{d\tau}M=-\frac{\partial}{\partial\xi}h_0=-\frac{\partial}{\partial\xi}H_0=0, \tag{7b}$$

$$\frac{d}{d\tau}\zeta=\frac{\partial}{\partial P}h_0=\frac{\partial}{\partial P}H_0, \tag{7c}$$

$$\frac{d}{d\tau}P=-\frac{\partial}{\partial\zeta}h_0=-\frac{\partial}{\partial\zeta}H_0=0, \tag{7d}$$

$$\frac{d}{d\tau}\theta=\frac{\partial}{\partial P_\theta}h_0=\frac{\partial}{\partial P_\theta}H_0, \tag{7e}$$

$$\frac{d}{d\tau}P_\theta=-\frac{\partial}{\partial\theta}h_0=-\frac{\partial}{\partial\theta}H_0, \tag{7f}$$

$$\frac{d}{d\tau}t=\frac{\partial}{\partial(-U)}h_0=1, \tag{7g}$$

$$\frac{d}{d\tau}(-U)=-\frac{\partial}{\partial t}h_0=-\frac{\partial}{\partial t}H_0=0. \tag{7h}$$

Note that $(-U,t)$ are a pair of conjugate variables. Clearly, it is convenient to set $\tau=t$ and interpret U as the total energy of the particle [16]. Note that the three invariants of motion are associated with three symmetries in the system. Conservation of M results from the gyrosymmetry, ξ independence of H_0; conservation of P results from the toroidal symmetry, ζ independence of H_0; conservation of U results from the temporal symmetry, t independence of H_0.

To close this section, we remind the readers that this formulation automatically guarantees the Hamiltonian characters of the guiding-center motion.

B. Lie-transform Hamiltonian perturbation method

The Lie-transform perturbation method is discussed in detail in Refs. [15–19]. Here we simply write down the formulas that we are going to use in this paper. Assuming that the perturbed one-form can be expanded in powers of ε_δ—a small parameter, up to $O(\varepsilon_\delta)$ we can transform the one-form according to the rule

$$\overline{\Gamma}_0=\Gamma_0+dS_0, \tag{8a}$$

$$\overline{\Gamma}_1=\Gamma_1-L_1\Gamma_0+dS_1, \tag{8b}$$

$$\overline{\Gamma}=\overline{\Gamma}_0+\overline{\Gamma}_1+O(\varepsilon_\delta^2), \tag{8c}$$

$$\Gamma=\Gamma_0+\Gamma_1+O(\varepsilon_\delta^2), \tag{8d}$$

where $L_1\Gamma_0$ is a one-form that is given by

$$(L_1\Gamma_0)_i=G_1^j\left(\frac{\partial\Gamma_{0i}}{\partial z^j}-\frac{\partial\Gamma_{0j}}{\partial z^i}\right), \tag{9}$$

with G_1^j known as the Lie-transform generating vector. The phase space variables are transformed by

$$\overline{z^i}=z^i+G_1^i+O(\varepsilon_\delta^2). \tag{10}$$

And a scalar is transformed in the way

$$\overline{f}=f-G_1^i\partial f/\partial z^i+O(\varepsilon_\delta^2). \tag{11}$$

To close this section, we emphasized that the Lie transform provides the relationships of functions, therefore, in practical calculation, both sides of each formula presented here should be evaluated with the same arguments [19].

III. CANONICAL VARIABLES FOR GYROCENTER MOTION

In this section, we shall find the canonical transform from the guiding-center coordinates to the gyrocenter coordinates.

Introduce the perturbation of the vector potential and the scalar potential written in the general form

$$\delta\mathbf{A}=\delta A_\psi(\mathbf{r},t)\nabla\psi+\delta A_\theta(\mathbf{r},t)\nabla\theta+\delta A_\zeta(\mathbf{r},t)\nabla\zeta, \tag{12a}$$

$$\delta\Phi=\delta\Phi(\mathbf{r},t), \tag{12b}$$

where $\mathbf{r}=\mathbf{R}+\boldsymbol{\rho}$ is the particle position, with $\mathbf{R}$ the guiding-center position and $\boldsymbol{\rho}$ the vector Larmor radius.

At this point, we clarify the orderings used in this paper. There are two independent small parameters: the ratio of the Larmor radius to the scale length of the equilibrium magnetic field, ε_B, and the ratio of perturbed field to the equilibrium field, ε_δ.

The perturbed one-form is written as

$$\Gamma=\Gamma_0+\Gamma_1, \tag{13}$$

with Γ_0 given in the previous section. Here and throughout this paper, subscript 0 denotes $O(\varepsilon_\delta^0)$ and subscript 1 denotes $O(\varepsilon_\delta)$. We are at the position to emphasize that [17] Γ_0 is kept up to $O(\varepsilon_B)$.

Γ_1 is given by

$$\begin{aligned}\Gamma_1&=\delta\mathbf{A}\cdot d(\mathbf{R}+\boldsymbol{\rho})-\delta\Phi\,dt\\&=(\delta A_\psi+\delta\mathbf{A}\cdot\partial_\psi\boldsymbol{\rho})d\psi+(\delta A_\theta+\delta\mathbf{A}\cdot\partial_\theta\boldsymbol{\rho})d\theta\\&\quad+(\delta A_\zeta+\delta\mathbf{A}\cdot\partial_\zeta\boldsymbol{\rho})d\zeta+(\delta\mathbf{A}\cdot\partial_M\boldsymbol{\rho})dM\\&\quad+(\delta\mathbf{A}\cdot\partial_\xi\boldsymbol{\rho})d\xi-\delta\Phi\,dt\end{aligned} \tag{14a}$$

$$\begin{aligned}&\equiv\delta A_\psi^*\,d\psi+\delta A_\theta^*\,d\theta+\delta A_\zeta^*\,d\zeta+(\delta\mathbf{A}\cdot\partial_M\boldsymbol{\rho})dM\\&\quad+(\delta\mathbf{A}\cdot\partial_\xi\boldsymbol{\rho})d\xi-\delta\Phi\,dt.\end{aligned} \tag{14b}$$

Note that we have retained in Eq. (14b) the Δ term defined by

$$\begin{aligned}\Delta&\equiv(\delta A_\psi^*\,d\psi+\delta A_\theta^*\,d\theta+\delta A_\zeta^*\,d\zeta)\\&\quad-(\delta A_\psi\,d\psi+\delta A_\theta\,d\theta+\delta A_\zeta\,d\zeta)\end{aligned} \tag{15a}$$

$$\equiv\delta\mathbf{A}\cdot\partial_\psi\boldsymbol{\rho}\,d\psi+\delta\mathbf{A}\cdot\partial_\theta\boldsymbol{\rho}\,d\theta+\delta\mathbf{A}\cdot\partial_\zeta\boldsymbol{\rho}\,d\zeta, \tag{15b}$$

Equation (14b) can be put in the form

$$\Gamma_1 = \delta A_\psi^* \left(\frac{\partial \psi}{\partial P_\theta} dP_\theta + \frac{\partial \psi}{\partial P} dP \right) + \delta A_\theta^* \, d\theta + \delta A_\zeta^* \, d\zeta + (\delta \mathbf{A} \cdot \partial_M \boldsymbol{\rho}) dM + (\delta \mathbf{A} \cdot \partial_\xi \boldsymbol{\rho}) d\xi - \delta \Phi \, dt. \tag{16}$$

Now we can carry out the Lie transform up to the first order in ε_δ. To do this, we choose

$$dS_0 = 0, \tag{17}$$

and consequently

$$\overline{\Gamma}_0 = \Gamma_0 + O(\varepsilon_B^2). \tag{18}$$

In Eq. (18), we have explicitly indicated that $\overline{\Gamma}_0$ and Γ_0 are kept up to $O(\varepsilon_B)$. Since we do not want to transform t and the system is independent of τ, we set

$$G_1^t = 0 = \partial S / \partial U, \tag{19a}$$

$$G_1^\tau = 0 = \partial S / \partial \tau. \tag{19b}$$

We shall make a symplectic transform to transfer all of the perturbations to Hamiltonian. To do this, we have the generating vector

$$G_1^\xi = -\left(\delta \mathbf{A} \cdot \frac{\partial \boldsymbol{\rho}}{\partial M} + \frac{\partial S_1}{\partial M} \right), \tag{20a}$$

$$G_1^\zeta = -\left(\delta A_\psi^* + \frac{\partial S_1}{\partial \psi} \right) \frac{\partial \psi}{\partial P}, \tag{20b}$$

$$G_1^\theta = -\left(\delta A_\psi^* + \frac{\partial S_1}{\partial \psi} \right) \frac{\partial \psi}{\partial P_\theta}, \tag{20c}$$

$$G_1^M = \delta \mathbf{A} \cdot \frac{\partial \boldsymbol{\rho}}{\partial \xi} + \frac{\partial S_1}{\partial \xi}, \tag{20d}$$

$$G_1^P = \delta A_\zeta^* + \frac{\partial S_1}{\partial \zeta}, \tag{20e}$$

$$G_1^{P_\theta} = \delta A_\theta^* + \frac{\partial S_1}{\partial \theta}, \tag{20f}$$

$$G_1^U = \delta \Phi - \frac{\partial S_1}{\partial t}. \tag{20g}$$

And the transformed first order (in ε_δ) one-form is given by

$$\overline{\Gamma}_1 = -\overline{H}_1 \, d\tau, \tag{21a}$$

$$\overline{H}_1 = \delta\varphi - (\partial_t + \dot{\psi}_0 \partial_\psi + \dot{\theta}_0 \partial_\theta + \dot{\zeta}_0 \partial_\zeta + \dot{\xi}_0 \partial_\xi) S_1, \tag{21b}$$

$$\delta\varphi \equiv \delta\Phi - \dot{\psi}_0 \delta A_\psi^* - \dot{\theta}_0 \delta A_\theta^* - \dot{\zeta}_0 \delta A_\zeta^* - \dot{\xi}_0 \delta \mathbf{A} \cdot \partial_\xi \boldsymbol{\rho}, \tag{21c}$$

where the overdot means d/dt, and

$$\dot{\psi}_0 = -\partial H_0 / \partial \theta (\partial \psi / \partial P_\theta), \tag{22a}$$

$$\dot{\theta}_0 = \partial H_0 / \partial P_\theta, \tag{22b}$$

$$\dot{\zeta}_0 = \partial H_0 / \partial P, \tag{22c}$$

$$\dot{\xi}_0 = \partial H_0 / \partial M. \tag{22d}$$

Note that every term in Eq. (22) is independent of (ξ, ζ, t). To make the transformed one-form independent of the gyrophase, we choose

$$\overline{H}_1 = \langle \delta\varphi \rangle \equiv \langle \delta\Phi \rangle - (\dot{\psi}_0 \langle \delta A_\psi^* \rangle + \dot{\theta}_0 \langle \delta A_\theta^* \rangle + \dot{\zeta}_0 \langle \delta A_\zeta^* \rangle + \dot{\xi}_0 \langle \delta \mathbf{A} \cdot \partial_\xi \boldsymbol{\rho} \rangle), \tag{23}$$

where $\langle \cdots \rangle$ denotes gyroaveraging.

It is not hard to recognize that

$$\dot{\xi}_0 \langle \delta \mathbf{A} \cdot \partial_\xi \boldsymbol{\rho} \rangle \sim O(\varepsilon_B^0). \tag{24}$$

To understand Eq. (24), it is helpful to have a look at Eq. (38c). Now we can examine the contribution of the Δ term to $\overline{H}_1$. Using $\dot{\theta}_0 / \dot{\xi}_0 \sim O(\varepsilon_B)$, $(\delta \mathbf{A} \cdot \partial_\theta \boldsymbol{\rho}) / (\delta \mathbf{A} \cdot \partial_\xi \boldsymbol{\rho}) \sim O(\varepsilon_B)$ and similar equations, one can find that

$$\frac{\langle \dot{\psi}_0 \delta \mathbf{A} \cdot \partial_\psi \boldsymbol{\rho} + \dot{\theta}_0 \delta \mathbf{A} \cdot \partial_\theta \boldsymbol{\rho} + \dot{\zeta}_0 \delta \mathbf{A} \cdot \partial_\zeta \boldsymbol{\rho} \rangle}{\dot{\xi}_0 \langle \delta \mathbf{A} \cdot \partial_\xi \boldsymbol{\rho} \rangle} \sim O(\varepsilon_B^2). \tag{25}$$

Combining Eqs. (24) and (25), we conclude that the contribution of the Δ term to $\overline{H}_1$ and $\overline{\Gamma}_1$ is $\sim O(\varepsilon_B^2)$. However, $\overline{\Gamma}_0$ has only been kept to $O(\varepsilon_B)$ [see, Eq. (18)]. Clearly the Δ term cannot be self-consistently included, it has to be dropped. We shall set

$$(\delta A_\psi^*, \delta A_\theta^*, \delta A_\zeta^*) = (\delta A_\psi, \delta A_\theta, \delta A_\zeta). \tag{26}$$

The final gyroaveraged first order (in ε_δ) Hamiltonian is written as

$$\overline{H}_1 = \langle \delta\varphi \rangle \equiv \langle \delta\Phi \rangle - (\dot{\psi}_0 \langle \delta A_\psi \rangle + \dot{\theta}_0 \langle \delta A_\theta \rangle + \dot{\zeta}_0 \langle \delta A_\zeta \rangle + \dot{\xi}_0 \langle \delta \mathbf{A} \cdot \partial_\xi \boldsymbol{\rho} \rangle). \tag{27}$$

Note that the discussion on the reason why the Δ term should be dropped has been similarly made in Ref. [11] in deriving the final first order Lagrangian for DKE. The equations of motion are readily found,

$$d\overline{M}/dt = 0, \tag{28a}$$

$$d\overline{U}/dt = \partial_t \overline{H}_1, \tag{28b}$$

$$d\overline{P}/dt = -\partial_{\overline{\zeta}} \overline{H}_1. \tag{28c}$$

Accordingly, we have the governing equation for S_1, the gauge function of the Hamiltonian Lie transform,

$$\widetilde{\delta\varphi} = (\partial_t + \dot{\psi}_0 \partial_\psi + \dot{\theta}_0 \partial_\theta + \dot{\zeta}_0 \partial_\zeta + \dot{\xi}_0 \partial_\xi) S_1, \tag{29}$$

where $\widetilde{\delta\varphi} \equiv \delta\varphi - \langle \delta\varphi \rangle$.

Following the previous gyrokinetic theory [4,7], we temporarily approximate the solution to Eq. (29) as

$$\dot{\xi}_0 \partial_\xi S_1 \approx \widetilde{\delta\varphi}, \tag{30}$$

which indicates that

$$\partial_t S_1 \sim \dot{\psi}_0 \partial_\psi S_1 \sim \dot{\theta}_0 \partial_\theta S_1 \sim \dot{\zeta}_0 \partial_\zeta S_1 \ll \dot{\xi}_0 \partial_\xi S_1 \sim \widetilde{\delta\Phi} \sim \dot{\psi}_0 \widetilde{\delta A}_\psi \sim \dot{\theta}_0 \widetilde{\delta A}_\theta \sim \dot{\zeta}_0 \widetilde{\delta A}_\zeta \sim \dot{\xi}_0 \widetilde{\delta \mathbf{A} \cdot \partial_\xi \boldsymbol{\rho}}. \tag{31}$$

The essence of this approximation will be made clear in Sec. VII.

With the help of Eq. (26) and the approximate solution to Eq. (29) [Eqs. (30–31)], we obtained

$$G_1^M \approx \langle \delta \mathbf{A} \cdot \partial_\xi \boldsymbol{\rho} \rangle + (\widetilde{\delta\Phi} - \dot{\psi}_0 \widetilde{\delta A}_\psi - \dot{\theta}_0 \widetilde{\delta A}_\theta - \dot{\zeta}_0 \widetilde{\delta A}_\zeta)/B, \tag{32a}$$

$$G_1^U \approx \delta\Phi, \tag{32b}$$

$$G_1^P \approx \delta A_\zeta. \tag{32c}$$

The other components of generating vector and equation of motion will not be used in the following discussions, therefore we do not present them here.

We have retained in Eqs. (27) and (32) the drift terms that were previously neglected [1–10]. To make it clear, we write

$$\begin{aligned}\bar{H}_1 &= \langle \delta\Phi \rangle - (\dot{\psi}_0 \langle \delta A_\psi \rangle + \dot{\theta}_0 \langle \delta A_\theta \rangle + \dot{\zeta}_0 \langle \delta A_\zeta \rangle + \dot{\xi}_0 \langle \delta \mathbf{A} \cdot \partial_\xi \boldsymbol{\rho} \rangle) \\ &= \langle \delta\Phi - V_\parallel \delta A_\parallel - \dot{\xi}_0 \delta \mathbf{A} \cdot \partial_\xi \boldsymbol{\rho} \rangle - \mathbf{V}_d \cdot \langle \delta \mathbf{A}_\perp \rangle,\end{aligned} \tag{33}$$

with $\delta\mathbf{A} = \delta A_\parallel \mathbf{b} + \delta\mathbf{A}_\perp$, $\delta A_\parallel = \mathbf{b}\cdot\delta\mathbf{A}$; $V_\parallel \mathbf{b} + \mathbf{V}_d = d\mathbf{R}/dt$, $V_\parallel = \mathbf{b}\cdot d\mathbf{R}/dt$; $\mathbf{b} = \mathbf{B}/B$. Note that although $\mathbf{V}_d \sim O(\varepsilon_B)$, we have $\mathbf{V}_d \cdot \delta\mathbf{A}_\perp \sim O(\varepsilon_B^0)$, since $\delta\mathbf{A}_\perp \sim O(\varepsilon_B^{-1})$. Therefore, to be consistent with the ordering of $\bar{\Gamma}_0$ in Eq. (18), the $\mathbf{V}_d \cdot \delta\mathbf{A}_\perp$ term should be retained in $\bar{H}_1$.

To close this section, we point out that, the Δ term can be systematically dropped without changing the Hamiltonian character of the dynamics, while the $\mathbf{V}_d \cdot \delta\mathbf{A}_\perp$ term cannot be. More discussions on this term will be made in the following.

IV. NEW REPRESENTATION OF GKE

Before deriving the GKE, we point out that among the eight variables used in the extended phase space, there are only six independent physical variables as was discussed in Ref. [16]. To derive the GKE, instead of using the coordinates $(\bar{M}, \bar{P}_\theta, \bar{P}, \bar{\xi}, \bar{\theta}, \bar{\zeta}; t)$ we shall use the coordinates $(\bar{M}, \bar{U}, \bar{P}, \bar{\xi}, \bar{\theta}, \bar{\zeta}; t)$. The Vlasov equation in these coordinates can be written as

$$\frac{\partial \bar{f}}{\partial t} + \frac{d\bar{M}}{dt}\frac{\partial \bar{f}}{\partial \bar{M}} + \frac{d\bar{U}}{dt}\frac{\partial \bar{f}}{\partial \bar{U}} + \frac{d\bar{P}}{dt}\frac{\partial \bar{f}}{\partial \bar{P}} + \frac{d\bar{\xi}}{dt}\frac{\partial \bar{f}}{\partial \bar{\xi}} + \frac{d\bar{\theta}}{dt}\frac{\partial \bar{f}}{\partial \bar{\theta}} + \frac{d\bar{\zeta}}{dt}\frac{\partial \bar{f}}{\partial \bar{\zeta}} = 0. \tag{34}$$

It is well known that the distribution function in terms of gyrocenter coordinates, $\bar{f}$, is independent of the gyrophase angle $\bar{\xi}$ [7,17]. Therefore we have

$$\frac{\partial \bar{f}}{\partial t} + \frac{d\bar{U}}{dt}\frac{\partial \bar{f}}{\partial \bar{U}} + \frac{d\bar{P}}{dt}\frac{\partial \bar{f}}{\partial \bar{P}} + \frac{d\bar{\theta}}{dt}\frac{\partial \bar{f}}{\partial \bar{\theta}} + \frac{d\bar{\zeta}}{dt}\frac{\partial \bar{f}}{\partial \bar{\zeta}} = 0, \tag{35}$$

where Eq. (28a) was used. Equations (28b,c) indicate that $d\bar{U}/dt = 0 + O(\varepsilon_\delta)$ and $d\bar{P} = 0 + O(\varepsilon_\delta)$. Therefore, expanding in terms of ε_δ, we have

$$\bar{f} = \bar{f}_0(\bar{M}, \bar{U}, \bar{P}) + \delta\bar{f}(\bar{M}, \bar{U}, \bar{P}, \bar{\theta}, \bar{\zeta}, t), \tag{36a}$$

$$\left(\frac{d}{dt}\right)_0 \delta\bar{f} = -(\partial_{\bar{U}} \bar{f}_0 \partial_t - \partial_{\bar{P}} \bar{f}_0 \partial_{\bar{\zeta}}) \bar{H}_1, \tag{36b}$$

$$\left(\frac{d}{dt}\right)_0 = \frac{\partial}{\partial t} + \left(\frac{d\bar{\theta}}{dt}\right)_0 \frac{\partial}{\partial \bar{\theta}} + \left(\frac{d\bar{\zeta}}{dt}\right)_0 \frac{\partial}{\partial \bar{\zeta}}. \tag{36c}$$

Transforming back to guiding-center coordinates, we have up to $O(\varepsilon_\delta)$

$$f = f_0(M, U, P) + G_1^M \frac{\partial f_0}{\partial M} + G_1^U \frac{\partial f_0}{\partial U} + G_1^P \frac{\partial f_0}{\partial P} + \delta f, \tag{37a}$$

$$\frac{d}{dt}\delta f = -(\partial_U f_0 \partial_t - \partial_P f_0 \partial_\zeta)(\langle \delta\Phi \rangle - \dot{\psi}_0 \langle \delta A_\psi \rangle - \dot{\theta}_0 \langle \delta A_\theta \rangle - \dot{\zeta}_0 \langle \delta A_\zeta \rangle - \dot{\xi}_0 \langle \delta\mathbf{A} \cdot \partial_\xi \boldsymbol{\rho} \rangle), \tag{37b}$$

where d/dt is the time rate of change evaluated along the unperturbed guiding-center orbit.

Note that Eq. (37) can be used for electromagnetic modes with arbitrary toroidal mode number *n*. It clearly reveals the effects of temporal and toroidal symmetry breaking. Every term in these equations has its own unambiguous physical interpretation. The adiabatic part of the perturbed distribution function in Eq. (37a) is completely determined by the perturbations of the magnetic moment, the total energy, and the canonical toroidal momentum; the coefficients of $\partial f_0/\partial M$, $\partial f_0/\partial U$, and $\partial f_0/\partial P$ are the perturbations of *M*, *U*, and *P*, respectively. The nonadiabatic part of the distribution function, δf, is purely due to the nonconservation of the total energy and the nonconservation of the canonical toroidal momentum introduced by the perturbation of fields. It is clearly shown that, within the frame of linear gyrokinetic theory, the free energy associated with $\partial_P f_0$ (M, U, P) does not nonadiabatically drive any axisymmetric modes.

V. LIMITING CASES

In order to compare the new representation of GKE with the existing theory, we present three limiting cases in this section. For readers who are not interested in the details of comparison, this section may be skipped.

A. Eikonal form

To explicitly evaluate the gyroaveraging used in the previous section, we introduce the eikonal ansatz following the conventional way [1,14], $(\delta\Phi,\delta\mathbf{A})\sim\exp(i\mathbf{k}_\perp\cdot\boldsymbol{\rho})$, with $\mathbf{k}_\perp$ being the perpendicular component of the wave vector. Then it is merely a simple mathematical exercise to verify that

$$\langle\delta\Phi\rangle=J_0(k_\perp\rho)\,\delta\Phi_c, \tag{38a}$$

$$\langle\delta A_\zeta\rangle=J_0(k_\perp\rho)\,\delta A_{\zeta c}, \tag{38b}$$

$$\langle\delta\mathbf{A}\cdot\partial_\xi\boldsymbol{\rho}\rangle=-\frac{J_1(k_\perp\rho)}{k_\perp\rho/2}M\,\delta B_c, \tag{38c}$$

where J_0,J_1 are the zeroth order and the first order Bessel functions, respectively; $\delta B=\mathbf{b}\cdot\nabla\times\delta\mathbf{A}$; the subscript c means the corresponding quantity is evaluated at the guiding-center position.

Now we can write the new representation of GKE in the eikonal form as

$$\begin{aligned}f={}&f_0(M,U,P)+\delta f+\delta\Phi\frac{\partial f_0}{\partial U}+\delta A_\zeta\frac{\partial f_0}{\partial P}\\&+\frac{1}{B}\bigg[-\frac{J_1(k_\perp\rho)}{k_\perp\rho/2}M\,\delta B_c+(\delta\Phi-J_0\,\delta\Phi_c)\\&-\frac{d\mathbf{R}}{dt}\cdot(\delta\mathbf{A}-J_0\,\delta\mathbf{A}_c)\bigg]\frac{\partial f_0}{\partial M},\end{aligned} \tag{39a}$$

$$\begin{aligned}\frac{d}{dt}\delta f={}&-(\partial_U f_0\partial_t-\partial_P f_0\partial_\zeta)\bigg[J_0\bigg(\delta\Phi_c-\frac{d\mathbf{R}}{dt}\cdot\delta\mathbf{A}_c\bigg)\\&+\frac{J_1(k_\perp\rho)}{k_\perp\rho/2}M\,\delta B_c\bigg],\end{aligned} \tag{39b}$$

with d/dt evaluated along the unperturbed guiding-center orbit. In obtaining Eq. (39), we have used Eq. (32).

To compare Eq. (39) with Ref. [10], we need some lengthy mathematical manipulations, which are briefly summarized as follows. First, we write down the relevant equations in Ref. [10] as

$$\begin{aligned}f={}&f_0+\delta\hat{f}+\bigg[\frac{\partial f_0}{\partial U}+\frac{\partial f_0}{\partial M}\frac{1}{B}\bigg(1+\frac{i}{\omega}v_\parallel\mathbf{b}\cdot\nabla\bigg)+\frac{ig}{B\omega}\frac{\partial f_0}{\partial P_\zeta}\mathbf{b}\cdot\nabla\bigg]\\&\times(\delta\Phi-J_0\,\delta\Phi_c)-\frac{\partial f_0}{\partial M}\frac{1}{B}\frac{J_1(k_\perp\rho)}{k_\perp\rho/2}M\,\delta B_c\\&-\frac{\nabla P_\zeta\times\mathbf{b}}{B}\cdot\delta\mathbf{A}\frac{\partial f_0}{\partial P_\zeta}-\frac{i}{B\omega}\frac{\partial f_0}{\partial P_\zeta}\nabla\psi\times\mathbf{b}\cdot\nabla J_0\,\delta\Phi_c,\end{aligned} \tag{40a}$$

$$\frac{d}{dt}\delta\hat{f}=\frac{\partial f_0}{\partial U}\bigg(1-\frac{\omega_*}{\omega}\bigg)\hat{X}, \tag{40b}$$

$$\omega_*=-i\frac{\partial f_0/\partial P_\zeta}{\partial f_0/\partial U}\partial_\zeta, \tag{40c}$$

$$\hat{X}=\bigg(\frac{d\mathbf{R}}{dt}-v_\parallel\mathbf{b}\bigg)\cdot\nabla J_0\,\delta\Phi_c+i\omega\frac{J_1(k_\perp\rho)}{k_\perp\rho/2}M\,\delta B_c. \tag{40d}$$

Note that a few of the misprints in Ref. [10] [Eqs. (12)–(14) there] have been corrected. Note also that $P_\zeta=-P$, which is due to the different definitions used in Ref. [10] and in this paper. Using $\mathbf{B}=g\nabla\zeta+\nabla\zeta\times\nabla\psi$, one can show that the three terms involving $\partial f_0/\partial P_\zeta$ in Eq. (40a) can be put into the following form:

$$\frac{\partial f_0}{\partial P_\zeta}\bigg(-\frac{\nabla P_\zeta\times\mathbf{b}}{B}\cdot\delta\mathbf{A}+\frac{ig}{B\omega}\mathbf{b}\cdot\nabla\,\delta\Phi-\frac{i}{\omega}\partial_\zeta J_0\,\delta\Phi_c\bigg). \tag{41}$$

Using $\partial_t=-i\omega$ and $-\mathbf{b}\cdot\nabla\,\delta\Phi+i\omega\,\delta A_\parallel=0$ in accordance with Ref. [10] to restore ∂_t and $\delta A_\parallel$, redefining the nonadiabatic part of the perturbed distribution function as

$$\delta f=\delta\hat{f}-\frac{\partial f_0}{\partial U}\bigg(1-\frac{\omega_*}{\omega}\bigg)J_0\,\delta\Phi_c, \tag{42}$$

and finally using P instead of P_ζ [changing the signs of ω_* and the last two terms in the bracket in Eq. (41)], we obtained

$$\begin{aligned}f={}&f_0+\delta f+\frac{\partial f_0}{\partial U}\delta\Phi+\frac{\partial f_0}{\partial P}\bigg(\frac{g}{B}\delta A_\parallel+\frac{\delta\mathbf{A}\times\mathbf{b}}{B}\cdot\nabla P\bigg)\\&+\frac{\partial f_0}{\partial M}\frac{1}{B}\bigg[-\frac{J_1(k_\perp\rho)}{k_\perp\rho/2}M\,\delta B_c+(\delta\Phi-J_0\,\delta\Phi_c)\\&-v_\parallel(\delta A_\parallel-\delta A_{\parallel c})\bigg],\end{aligned} \tag{43a}$$

$$\begin{aligned}\frac{d}{dt}\delta f={}&-(\partial_U f_0\partial_t-\partial_P f_0\partial_\zeta)\bigg[J_0(\delta\Phi_c-v_\parallel\,\delta A_\parallel)\\&+\frac{J_1(k_\perp\rho)}{k_\perp\rho/2}M\,\delta B_c\bigg].\end{aligned} \tag{43b}$$

Using $\nabla P=-\nabla\psi+O(\varepsilon_B)$, it is easy to show that

$$\frac{g}{B}\delta A_\parallel+\frac{\delta\mathbf{A}\times\mathbf{b}}{B}\cdot\nabla P=\delta A_\zeta. \tag{44}$$

Substituting Eq. (44) into Eq. (43a), we obtained the resulting equations, which agree well with Eq. (39). The only difference is that we have kept the $\mathbf{V}_d\cdot\delta\mathbf{A}_{\perp c}$ and $\mathbf{V}_d\cdot\delta\mathbf{A}_\perp$ terms in Eq. (39). As we have pointed out, these two terms are not included in Ref. [1], which is the starting point of Ref. [10]. More discussions on this point will be given in the next section.

B. DKE form

Dropping the effects of FLR, we can obtain the DKE. This can be done by setting

$$J_0(k_\perp\rho)=1, \tag{45a}$$

$$\frac{J_1(k_\perp \rho)}{k_\perp \rho/2} = 1, \tag{45b}$$

$$\delta\Phi = \delta\Phi_c, \tag{45c}$$

$$\delta\mathbf{A} = \delta\mathbf{A}_c. \tag{45d}$$

The resulting DKE reads

$$f = f_0 + \delta f + \frac{\partial f_0}{\partial U}\delta\Phi - \frac{\partial f_0}{\partial M} M \frac{\delta B}{B} + \frac{\partial f_0}{\partial P}\delta A_\zeta, \tag{46a}$$

$$\frac{d}{dt}\delta f = -(\partial_U f_0 \partial_t - \partial_P f_0 \partial_\zeta)\left(\delta\Phi - \frac{d\mathbf{R}}{dt}\cdot\delta\mathbf{A} + M\,\delta B\right), \tag{46b}$$

which agrees with Ref. [11]. This representation of GKE exactly recovers, in dropping the effects of FLR, the DKE [11] including the $\mathbf{V}_d \cdot \delta\mathbf{A}_\perp$ term.

C. Small-banana-width limit

Finally, it may be useful to present the small-banana-width limit of the new representation of GKE. Dropping the effects of FBW in Eq. (37), we may make the replacement

$$[\partial_M, \partial_U, \partial_P] f_0(M,U,P) \rightarrow [\partial_M, \partial_U, -\partial_\psi] f_0(M,U,\psi). \tag{47}$$

Note that a similar small-banana-width approximation has been made and discussed in Ref. [11]. Then Eq. (37) reduces to

$$f = f_0(M,U,\psi) + \delta f + G_1^M \partial_M f_0 + G_1^U \partial_U f_0 - G_1^P \partial_\psi f_0, \tag{48a}$$

$$\frac{d}{dt}\delta f = -(\partial_U f_0 \partial_t + \partial_\psi f_0 \partial_\zeta)\langle\delta\varphi\rangle. \tag{48b}$$

In most practical cases, when the effects of FBW are dropped, the Maxwellian distribution $f_M(U,\psi)$ may be used as an approximate equilibrium thermal plasma distribution and the slowing-down distribution $f_S(M,U,\psi)$ may be used as an approximate equilibrium fast ion distribution. Since $\partial_U f_M \partial_t \langle\delta\varphi\rangle$ or $\partial_U f_S \partial_t \langle\delta\varphi\rangle$ is merely a damping term, as is well known, we may conclude that there is no nonadiabatic linear driving to any axisymmetric modes, with the Maxwellian distribution or the slowing-down distribution used as the equilibrium distribution.

To compare Eq. (48) with the existing literatures, we assume

$$\langle\delta\varphi\rangle \sim \exp(-i\omega t). \tag{49}$$

Then Eq. (48b) can be written as

$$\frac{d}{dt}\delta f = i(\omega - \omega_*)\partial_U f_0 \langle\delta\varphi\rangle, \tag{50a}$$

$$\omega_* = -i\frac{\partial_\psi f_0}{\partial_U f_0}\partial_\zeta. \tag{50b}$$

We are at the position to make some remarks on the ω_* term in the GKE. Conventionally, when using the local Maxwellian distribution $f_M(U,\psi)$ as the equilibrium thermal plasmas distribution or using the slowing-down distribution $f_S(M,U,\psi)$ as the equilibrium fast ions distribution, $\omega_* \propto \partial_\theta$ in the GKE [13,25]. However, recently, when retaining the effects of FBW, it is shown that $\omega_* \propto \partial_\zeta$ in GKE [10] or in DKE [11]. To clarify this important controversial issue, first we go back to Ref. [10]. Note that the classical GKE, Eq. (7) in Ref. [10], agrees with the modern GKE [Eq. (35) in Ref. [7]]. When substituting $f_M(U,\psi)$ or $f_S(M,U,\psi)$ as the equilibrium distribution in the GKE [Eq. (7) in Ref. [10] or Eq. (35) in Ref. [7]], one gets $\omega_* \propto \partial_\theta$, as in Refs. [13,25]. When substituting $f_0(M,U,P)$ as the equilibrium distribution in the GKE [Eq. (7) in Ref. [10]], one gets $\omega_* \propto \partial_\zeta$, as was shown in Ref. [10]. The point is that $f_M(U,\psi)$ or $f_S(M,U,\psi)$ is not a true equilibrium distribution in an axisymmetric torus, since it is not a constant of motion. Therefore, one should be careful when using $f_M(U,\psi)$ or $f_S(M,U,\psi)$ as the approximate equilibrium distribution. It is well-known that it is important to keep the Hamiltonian character of the motion in deriving the GKE. And the Hamiltonian character is clearly kept in deriving Eq. (37). Therefore, the ω term in the GKE is associated with the nonconservation of the gyrocenter energy and the ω_* term in the GKE is responsible for the nonconservation of the canonical toroidal momentum of the gyrocenter. Clearly $\omega_* \propto \partial_\zeta$ is related to the Hamiltonian character of the motion, it should not be violated. In substituting $f_M(U,\psi)$ or $f_S(M,U,\psi)$ as the equilibrium distribution in the GKE [Eq. (7) in Ref. [10] or Eq. (35) in Ref. [7]], the Hamiltonian character of the motion is lost, since neither $f_M(U,\psi)$ nor $f_S(M,U,\psi)$ satisfies the lowest order GKE for an axisymmetric torus. In obtaining Eq. (50) through making the replacement, Eq. (47) in Eq. (37), the Hamiltonian character of the motion is safely kept, since the $\omega_* \propto \partial_\zeta$ term in Eq. (50) is indeed related to the nonconservation of the canonical toroidal momentum of the gyrocenter introduced by the perturbation. It was pointed out by Littlejohn [17] that the GKE should be understood as a representation; similarly, we point out here that Eqs. (47) and (50) should be understood as a representation rather than an approximation.

To close this section, we point out that the incorrect $\omega_* \propto \partial_\theta$ in the literature [13,25] is not due to an error in the original GKE's [1–9]; it is due to the carelessness in using $f_M(U,\psi)$ as the approximate equilibrium distribution, as has been discussed in the above paragraph. However, the GKE's in Refs. [1–4,6–9] do not include the $\mathbf{V}_d \cdot \langle\delta\mathbf{A}_\perp\rangle$ term in the first order (in ε_δ) gyroaveraged Hamiltonian $\overline{H}_1$. The GKE in Ref. [5] has included the drift term induced by the equilibrium radial electric field, but it has not included the drift term induced by the inhomogeneity of the equilibrium magnetic field. And all of these GKE's are not invariant with respect to the gauge transformation of the perturbation fields in toroidal geometry, as will be discussed in detail in the next section.

VI. INVARIANCE WITH RESPECT TO GAUGE TRANSFORMATION OF PERTURBATION FIELDS

A. General remarks on gauge invariance

Now, we shall discuss the invariance with respect to gauge transformation of the perturbation fields.

It is well known that the gauge transformation of the vector potential and the scalar potential of the electromagnetic fields does not change electromagnetic fields themselves and does not change the Hamiltonian dynamics. This is very clear in Hamiltonian formalism, since the gauge transformation only adds a total differential to the fundamental one-form and adding an arbitrary total differential to the fundamental one-form does not change the Hamiltonian dynamics. However, at first sight, it seems that there might be some problems in making our formalism gauge invariant. Since a general gauge transformation introduces a time-dependent component to the scalar potential, the unperturbed Hamiltonian H_0 becomes time dependent; it seems that we lose the three important constants of motion. Of course, this is not the case; one can verify that the three constants of motion are still there. However, this clearly makes the unperturbed system formally time dependent and unnecessarily makes the problem complicated. This unnecessary complication can be avoided by the following scenario.

Since the system equilibrium state is physically time independent, we do not want to introduce any gauge transformation to make the unperturbed system mathematically or formally time dependent. We shall choose a specific gauge [23,24] to display the three constants of motion as clearly as in Sec. II A. We only allow gauge transformation of the perturbation fields. Given the perturbation fields

$$\delta\mathbf{B} = \boldsymbol{\nabla}\times\delta\mathbf{A}, \tag{51a}$$

$$\delta\mathbf{E} = -\partial_t\delta\mathbf{A} - \boldsymbol{\nabla}\,\delta\Phi, \tag{51b}$$

we make the following gauge transformation:

$$\delta\mathbf{A}' = \delta\mathbf{A} + \boldsymbol{\nabla}\,\delta g(\mathbf{r},t), \tag{52a}$$

$$\delta\Phi' = \delta\Phi - \partial_t\delta g(\mathbf{r},t). \tag{52b}$$

Note that $(\delta\mathbf{A},\delta\Phi)$ have already been written in a general form in Eq. (12). So ‘‘the GKE for $(\delta\mathbf{A}',\delta\Phi')$’’ can be directly written out by replacing $(\delta\mathbf{A},\delta\Phi)$ by $(\delta\mathbf{A}',\delta\Phi')$ in ‘‘the GKE for $(\delta\mathbf{A},\delta\Phi)$.’’

However, the GKE for $(\delta\mathbf{A}',\delta\Phi')$ may be obtained in a different way. With the gauge transformation of the perturbation potentials, the first order fundamental one-form given in Eq. (16) is changed to

$$\Gamma_1' = \Gamma_1 + d\mathbf{r}\cdot\boldsymbol{\nabla}\,\delta g + \partial_t\delta g(\mathbf{r},t)dt \tag{53a}$$

$$= \Gamma_1 + d\,\delta g. \tag{53b}$$

According to Eq. (8b), the Lie-transformed $\overline{\Gamma_1}$ is changed by the gauge transformation

$$\overline{\Gamma_1'} = \Gamma_1 - L_1\Gamma_0 + d(S_1 + \delta g) \tag{54a}$$

$$\equiv \Gamma_1 - L_1\Gamma_0 + dS_1'. \tag{54b}$$

Now, one can run the derivations described in Sec. III again. The only difference is that S_1 in Sec. III is symbolically changed to S_1'. The final results in Eqs. (27)–(29) and consequently Eq. (37) do not change, they are still determined by $(\delta\mathbf{A},\delta\Phi)$.

So, beginning with $(\delta\mathbf{A}',\delta\Phi')$, one may obtain the final GKE that is identical to the GKE for $(\delta\mathbf{A},\delta\Phi)$. Therefore, for one physical problem, we have two formally different GKE’s. However, from the above discussions, one can see that both the GKE’s follow the same Hamiltonian formalism. Therefore, the dynamics is the same. We may draw the following conclusion, which shall be referred to as an equivalence theorem.

If any $(\delta\mathbf{A}',\delta\Phi')$ can be related to $(\delta\mathbf{A},\delta\Phi)$ by the gauge transformation Eq. (52), then the GKE for $(\delta\mathbf{A}',\delta\Phi')$ is equivalent to the GKE for $(\delta\mathbf{A},\delta\Phi)$.

Now, for a given $(\delta\mathbf{A}',\delta\Phi')$, one may do a gauge transformation, Eq. (52), to reach $(\delta\mathbf{A},\delta\Phi)$. Equation (12) guarantees that there are no constraints on this gauge transformation, and that through the gauge transformation Eq. (52), any $(\delta\mathbf{A}',\delta\Phi')$ can be related to $(\delta\mathbf{A},\delta\Phi)$ represented in Eq. (12). The equivalence theorem guarantees that the GKE needed is exactly Eq. (37).

Therefore, we concluded that our GKE [Eq. (37)] does not depend on a specific gauge of the perturbation fields; it is invariant with respect to the gauge transformation of perturbation fields.

B. Importance of the $\mathbf{V}_d\cdot\delta\mathbf{A}_\perp$ term

Now, we can clarify why it is important to keep the $\mathbf{V}_d\cdot\delta\mathbf{A}_\perp$ term in the GKE.

Simply, when $\mathbf{V}_d=0$ or a gauge is chosen so that $\delta\mathbf{A}_\perp=0$, the $\mathbf{V}_d\cdot\delta\mathbf{A}_\perp$ term can be dropped out of the GKE. Generally $\mathbf{V}_d\neq 0$, so this term cannot be dropped for a gauge $\delta\mathbf{A}_\perp\neq 0$. The mathematical reason is that, although $\mathbf{V}_d\cdot\delta\mathbf{A}_\perp/V_\parallel\delta A_\parallel\sim O(\varepsilon_B)$ (when $|\delta\mathbf{A}_\perp|\sim\delta A_\parallel$), $\mathbf{V}_d\cdot\delta\mathbf{A}_\perp\sim O(\varepsilon_B^0)$ cannot be dropped since we have to keep any $O(\varepsilon_B^0)$ term in the fundamental one-form, as has been discussed in Sec. III. The physical reason is more fundamental; in dropping this term, one breaks the Hamiltonian structure of the dynamics and consequently loses the property of gauge invariance.

The GKE’s in Refs. [1–4,6–9] do not include the $\mathbf{V}_d\cdot\langle\delta\mathbf{A}_\perp\rangle$ term in the first order (in ε_δ) gyroaveraged Hamiltonian, $\overline{H_1}$; the GKE in Ref. [5] has included the drift term induced by the equilibrium radial electric field, but it has not included the drift term induced by the inhomogeneity of the equilibrium magnetic field. And all of these GKE’s are not invariant with respect to the gauge transformation of the perturbation fields in toroidal geometry.

To make this point clearer, we shall discuss two typical papers in more detail. One is typical of the classical GKE [1], the other is typical of the modern GKE [8]. In both these papers, the $\mathbf{V}_d \cdot \langle \delta \mathbf{A}_\perp \rangle$ has not been retained. In Ref. [1], it was explicitly indicated that the derivation does not depend on any specific gauge [see, notes after Eqs. (41) and (42) in Ref. [1]]; in Ref. [8], the authors did not indicate a specific gauge. To clearly show that these GKE's are not gauge invariant, we shall make an example.

It is well known that for ideal MHD shear Alfven wave perturbations, the perturbation fields can be described by two different gauges,

$$\delta \mathbf{B} = \nabla \times (\delta \mathbf{A}), \quad \delta \mathbf{A} = \delta A_\parallel \mathbf{b}, \tag{55a}$$

$$\delta \mathbf{E} = -\partial_t \delta A_\parallel \mathbf{b} - \nabla \delta \Phi; \tag{55b}$$

$$\mathbf{b} \cdot \delta \mathbf{E} = 0, \tag{55c}$$

$$\partial_t \vec{\xi}_\perp = \delta \mathbf{E} \times \mathbf{b} \tag{55d}$$

with $\vec{\xi}_\perp$ being the usual perpendicular component of the ideal MHD fluid displacement; and

$$\delta \mathbf{B} = \nabla \times (\vec{\xi}_\perp \times \mathbf{B}), \quad \delta \mathbf{A} = \vec{\xi}_\perp \times \mathbf{B}, \tag{56a}$$

$$\delta \mathbf{E} = -\partial_t (\vec{\xi}_\perp \times \mathbf{B}), \quad \delta \Phi = 0. \tag{56b}$$

In the well-known paper on fishbone modes [26] Chen, White, and Rosenbluth applied the GKE [1] with the gauge Eq. (55a,b) and they obtained the well-known result. However, if one used the gauge Eq. (56) in applying the GKE [1] to solve the same problem discussed in Ref. [26], one would not obtain the correct result. This is undoubtedly due to the fact that the GKE in Ref. [1] is not gauge invariant. It can be similarly verified that the GKE in Ref. [8] is not gauge invariant either.

With the effects of FLR ignored, it is straightforward to verify that our GKE gives same results with the two different gauges given in Eq. (55) and Eq. (56); the resonance interaction between the energetic trapped ions and the internal kink mode is proportional to $\mathbf{V}_d \cdot \delta \mathbf{E}_\perp$. In dropping the effects of FBW and FLR, our results obtained with the two different gauges agree with Ref. [26]. As we have shown that our GKE agrees with Ref. [1] when $\delta \mathbf{A}_\perp = 0$, the procedure to use Eq. (55) in our GKE is similar to Ref. [26]. It is similar to running from Eq. (43) back to Eq. (40), some mathematical manipulations that have not been displayed in Ref. [26]. Since we have shown that our GKE agrees with Ref. [11], the procedure to use Eq. (56) in our GKE, which is simpler than using Eq. (55), is similar to Ref. [11]. The only exception is that the initial condition $\Phi = 0$ used there is unnecessary in our case.

To close this section, we point out that even for ideal MHD shear Alfven wave perturbations, gauge invariance can provide considerable convenience. From our discussions above, one can see that using the gauge Eq. (56) is more convenient than using the gauge Eq. (55). It is also interesting to note that, for fishbone modes, the two different viewpoints ($\omega_* \propto \partial_\theta$ and $\omega_* \propto \partial_\zeta$ discussed in Sec. V C) give the same result [26,27], since $\vec{\xi}_\perp \sim \exp(i\theta - i\zeta)$.

C. Recovering the $\mathbf{V}_d \cdot \delta \mathbf{A}_\perp$ term

It is useful to show how to recover the $\mathbf{V}_d \cdot \langle \delta \mathbf{A}_\perp \rangle$ term in the first order (in ε_δ) gyro-averaged Hamiltonian within the frame of Refs. [1,8].

First, let us look at Ref. [8], which is a typical paper on the modern gyrokinetic theory.

We shall examine Eqs. (11) and (16–19) of Ref. [8]. In their Eq. (11), our Δ term was dropped by neglecting the inhomogeneity of equilibrium fields. In their Eqs. (16–18), our $\mathbf{V}_d \cdot \langle \delta \mathbf{A}_\perp \rangle$ term appeared. Note that $\mathbf{V}_d$ comes from the inhomogeneity of equilibrium fields. In their Eq. (19), the $\mathbf{V}_d \cdot \langle \delta \mathbf{A}_\perp \rangle$ term was dropped by neglecting the inhomogeneity of equilibrium fields. Clearly, the self-consistence of Ref. [8] in neglecting both the Δ term and the $\mathbf{V}_d \cdot \langle \delta \mathbf{A}_\perp \rangle$ term was related to neglecting the inhomogeneity of equilibrium fields. Note that $\mathbf{V}_d$ was still retained in the propagator in their GKE [see, their Eq. (41)]. As we have shown in Sec. III, it is unnecessary to neglect the inhomogeneity of equilibrium fields. It is due to the fact that the contribution of the Δ term to the gyroaveraged Hamiltonian is of $O(\varepsilon_B^2)$, which is irrelevant, that the Δ term can be systematically dropped. Therefore the $\mathbf{V}_d \cdot \langle \delta \mathbf{A}_\perp \rangle$ term can be self-consistently retained in Eq. (19) in Ref. [8].

Second, let us look at Ref. [1], which is a typical paper on the classical gyrokinetic theory.

We shall examine Eqs. (35–41) of Ref. [1]. It is not hard to recognize that the neglecting of the $\mathbf{V}_d \cdot \langle \delta \mathbf{A}_\perp \rangle$ term in their Eq. (41) is closely related to the third line of their Eq. (40). Therefore, we write down this key equation.

$$\langle \mathbf{v}_\perp \exp(iL) \rangle = (i v_\perp / k_\perp) J_1(k_\perp v_\perp / \Omega) \mathbf{k}_\perp \times \mathbf{b}, \tag{57}$$

with $L = \mathbf{k}_\perp \cdot \mathbf{b} \times \mathbf{v}$.

We observed that Eq. (57) implies that $v_\perp$ has been taken as a constant. This implies that

$$\langle \mathbf{v}_\perp \rangle = 0. \tag{58}$$

Note that $\mathbf{v}$ is the particle velocity. It is well known that Eq. (58) is true only to the zeroth order in ε_B. To the first order in ε_B, we have

$$\langle \mathbf{v}_\perp \rangle = \mathbf{V}_d. \tag{59}$$

This indicates that, to the first order in ε_B, we should use

$$\mathbf{v}_\perp = \mathbf{V}_d + \tilde{\mathbf{v}}_\perp, \tag{60a}$$

$$\langle \tilde{\mathbf{v}}_\perp \rangle = 0, \tag{60b}$$

$$|\tilde{\mathbf{v}}_\perp| \approx |\mathbf{v}_\perp| = v_\perp. \tag{60c}$$

Clearly, to first order in ε_B, Eq. (57) may be replaced by

$$\langle \mathbf{v}_\perp \exp(iL) \rangle = (i v_\perp / k_\perp) J_1(k_\perp v_\perp / \Omega) \mathbf{k}_\perp \times \mathbf{b} + \mathbf{V}_d J_0(k_\perp v_\perp / \Omega), \tag{61}$$

With this replacement, the $\mathbf{V}_d \cdot \langle \delta \mathbf{A}_\perp \rangle$ term can be easily recovered within the frame of Ref. [1].

To close this section, we point out that without the detailed comparison analysis discussed above, it is very easy to miss the $\mathbf{V}_d \cdot \langle \delta \mathbf{A}_\perp \rangle$ term in previous GKE's.

VII. GAUGE INVARIANT FORM OF GKE

To completely resolve the issue of gauge invariance, we have to directly verify that the perturbed distribution function in terms of the guiding-center coordinates f is invariant with respect to the gauge transformation. To determine the perturbed distribution f, the GKE, Eq. (37), is not complete; we need to determine (G_1^M, G_1^U, G_1^P), the generating vector of the Lie transform. At first, we use the approximate (G_1^M, G_1^U, G_1^P) given in Eq. (32), which is based on Eqs. (30) and (31) and equivalent to the previous theory [4,7]. In the last two sections, we have compared Eqs. (32) and (37) with previous GKE's in detail, and we have resolved the only difference, the $\mathbf{V}_d \cdot \delta \mathbf{A}_\perp$ term.

According to the general remarks on the gauge invariance made in the last section, we expected gauge invariance of the new GKE, combination of Eqs. (37) and (32). Now, we are at the position to verify the gauge invariance of the perturbed distribution function f in terms of guiding-center coordinates. Unfortunately, the results show that the gyrophase-dependent part of f in guiding-center coordinates is not gauge invariant. To resolve this problem, we first go back to reexamine the general remarks on the gauge invariance made in the last section. We understand that it is the formalism of the Hamiltonian Lie-transform that is gauge invariant. Therefore, if we had not introduced any approximations when deriving the GKE by the Hamiltonian Lie transform, we should have obtained a gauge invariant form of GKE. Guided by the general remarks on the gauge invariance, we recognized that it is important to have a proper solution to S_1 (known as the gauge function for the Hamiltonian Lie transform) [see Eqs. (54) and (29)]. Consequently, we have identified the problem, which lies in the approximation made in Eqs. (30)–(32). The problem is resolved as follows.

Using Helmholtz's theorem [28], we decompose the perturbation vector potential $\delta \mathbf{A}$ into two parts; one part $\delta \mathbf{A}^{\mathcal{R}}$ is rotational (vortex component) and the other part $\delta \mathbf{A}^{\mathcal{I}}$ is irrotational (source component),

$$\delta \mathbf{A}(\mathbf{r},t) = \delta \mathbf{A}^{\mathcal{R}}(\mathbf{r},t) + \delta \mathbf{A}^{\mathcal{I}}(\mathbf{r},t), \tag{62a}$$

$$\nabla \cdot \delta \mathbf{A}^{\mathcal{R}}(\mathbf{r},t) = 0, \tag{62b}$$

$$\delta \mathbf{A}^{\mathcal{I}}(\mathbf{r},t) = \nabla \delta\alpha(\mathbf{r},t). \tag{62c}$$

And the perturbation scalar potential $\delta\Phi$ is accordingly decomposed into two parts

$$\delta\Phi(\mathbf{r},t) = \delta\Phi^{\mathcal{R}}(\mathbf{r},t) + \delta\Phi^{\mathcal{I}}(\mathbf{r},t), \tag{63a}$$

$$\delta\Phi^{\mathcal{I}}(\mathbf{r},t) = -\partial_t \delta\alpha(\mathbf{r},t). \tag{63b}$$

Writing the perturbation vector potential in covariant representation, we have

$$\begin{aligned}\delta \mathbf{A}(\mathbf{r},t) &= \delta A_\psi(\mathbf{r},t)\nabla\psi + \delta A_\theta(\mathbf{r},t)\nabla\theta + \delta A_\zeta(\mathbf{r},t)\nabla\zeta \\ &\equiv [\delta A_\psi^{\mathcal{R}}(\mathbf{r},t) + \delta A_\psi^{\mathcal{I}}(\mathbf{r},t)]\nabla\psi + [\delta A_\theta^{\mathcal{R}}(\mathbf{r},t) \\ &\quad + \delta A_\theta^{\mathcal{I}}(\mathbf{r},t)]\nabla\theta + [\delta A_\zeta^{\mathcal{R}}(\mathbf{r},t) + \delta A_\zeta^{\mathcal{I}}(\mathbf{r},t)]\nabla\zeta,\end{aligned} \tag{64a}$$

$$\begin{aligned}\delta \mathbf{A}^{\mathcal{I}}(\mathbf{r},t) &= \delta A_\psi^{\mathcal{I}}(\mathbf{r},t)\nabla\psi + \delta A_\theta^{\mathcal{I}}(\mathbf{r},t)\nabla\theta + \delta A_\zeta^{\mathcal{I}}(\mathbf{r},t)\nabla\zeta \\ &\equiv [\partial_\psi \delta\alpha(\mathbf{r},t)]\nabla\psi + [\partial_\theta \delta\alpha(\mathbf{r},t)]\nabla\theta \\ &\quad + [\partial_\zeta \delta\alpha(\mathbf{r},t)]\nabla\zeta.\end{aligned} \tag{64b}$$

We shall also make the following decomposition:

$$\widetilde{\delta\varphi} = \widetilde{\delta\varphi}^{\mathcal{R}} + \widetilde{\delta\varphi}^{\mathcal{I}}, \tag{65a}$$

$$S_1 = S_1^{\mathcal{R}} + S_1^{\mathcal{I}}, \tag{65b}$$

$$\begin{pmatrix} G_1^M \\ G_1^U \\ G_1^P \end{pmatrix} = \begin{pmatrix} G_1^{M,\mathcal{R}} \\ G_1^{U,\mathcal{R}} \\ G_1^{P,\mathcal{R}} \end{pmatrix} + \begin{pmatrix} G_1^{M,\mathcal{I}} \\ G_1^{U,\mathcal{I}} \\ G_1^{P,\mathcal{I}} \end{pmatrix}, \tag{65c}$$

$$\widetilde{\delta\varphi}^{\mathcal{R}} = \widetilde{\delta\Phi}^{\mathcal{R}} - \dot\psi_0 \widetilde{\delta A_\psi}^{\mathcal{R}} - \dot\theta_0 \widetilde{\delta A_\theta}^{\mathcal{R}} - \dot\zeta_0 \widetilde{\delta A_\zeta}^{\mathcal{R}} - \dot\xi_0 \widetilde{\delta \mathbf{A}^{\mathcal{R}} \cdot \partial_\xi \boldsymbol{\rho}}, \tag{66a}$$

$$\widetilde{\delta\varphi}^{\mathcal{I}} = \widetilde{\delta\Phi}^{\mathcal{I}} - \dot\psi_0 \widetilde{\delta A}_\psi^{\mathcal{I}} - \dot\theta_0 \widetilde{\delta A}_\theta^{\mathcal{I}} - \dot\zeta_0 \widetilde{\delta A}_\zeta^{\mathcal{I}} - \dot\xi_0 \widetilde{\delta \mathbf{A}^{\mathcal{I}} \cdot \partial_\xi \boldsymbol{\rho}} \tag{66b}$$

$$= -(\partial_t + \dot\psi_0 \partial_\psi + \dot\theta_0 \partial_\theta + \dot\zeta_0 \partial_\zeta + \dot\xi_0 \partial_\xi)\widetilde{\delta\alpha}. \tag{66c}$$

In writing Eq. (66c), we have used Eqs. (63) and (64) and the relation [14]

$$(\partial_\xi \boldsymbol{\rho}) \cdot \nabla \delta\alpha = \partial_\xi \delta\alpha. \tag{67}$$

Substituting Eqs. (65a,b) and (66) into Eq. (29), we found

$$\begin{aligned}&(\partial_t + \dot\psi_0 \partial_\psi + \dot\theta_0 \partial_\theta + \dot\zeta_0 \partial_\zeta + \dot\xi_0 \partial_\xi)\begin{bmatrix} S_1^{\mathcal{R}} \\ S_1^{\mathcal{I}} \end{bmatrix} \\ &\quad = \begin{bmatrix} \widetilde{\delta\varphi}^{\mathcal{R}} \\ -(\partial_t + \dot\psi_0 \partial_\psi + \dot\theta_0 \partial_\theta + \dot\zeta_0 \partial_\zeta + \dot\xi_0 \partial_\xi)\widetilde{\delta\alpha} \end{bmatrix}.\end{aligned} \tag{68}$$

The solution to $S_1^{\mathcal{I}}$ is readily found

$$S_1^{\mathcal{I}} = -\widetilde{\delta\alpha}. \tag{69}$$

And the solution to $S_1^{\mathcal{R}}$ is obtained by following the previous approximation [4,7]

$$\dot\xi_0 \partial_\xi S_1^{\mathcal{R}} \approx \widetilde{\delta\varphi}^{\mathcal{R}} \tag{70a}$$

$$= \widetilde{\delta\Phi}^{\mathcal{R}} - \dot\psi_0 \widetilde{\delta A}_\psi^{\mathcal{R}} - \dot\theta_0 \widetilde{\delta A}_\theta^{\mathcal{R}} - \dot\zeta_0 \widetilde{\delta A}_\zeta^{\mathcal{R}} - \dot\xi_0 \widetilde{\delta \mathbf{A}^{\mathcal{R}} \cdot \partial_\xi \boldsymbol{\rho}}, \tag{70b}$$

$$\partial_t S_1^{\mathcal{R}} \sim \dot{\psi}_0 \partial_\psi S_1^{\mathcal{R}} \sim \dot{\theta}_0 \partial_\theta S_1^{\mathcal{R}} \sim \dot{\zeta}_0 \partial_\zeta S_1^{\mathcal{R}} \ll \dot{\xi}_0 \partial_\xi S_1^{\mathcal{R}} \sim \widetilde{\delta\Phi}^{\mathcal{R}} \sim \dot{\psi}_0 \widetilde{\delta A}_\psi^{\mathcal{R}} \sim \dot{\theta}_0 \widetilde{\delta A}_\theta^{\mathcal{R}} \sim \dot{\zeta}_0 \widetilde{\delta A}_\zeta^{\mathcal{R}} \sim \dot{\xi}_0 \widetilde{\delta \mathbf{A}^{\mathcal{R}} \cdot \partial_\xi \boldsymbol{\rho}}. \tag{71}$$

With the solution of S_1 obtained above, we found the generating vector for the Hamiltonian Lie transform,

$$G_1^{M,\mathcal{R}} \approx \langle \delta \mathbf{A}^{\mathcal{R}} \cdot \partial_\xi \boldsymbol{\rho} \rangle + \frac{1}{B} (\widetilde{\delta\Phi}^{\mathcal{R}} - \dot{\psi}_0 \widetilde{\delta A}_\psi^{\mathcal{R}} - \dot{\theta}_0 \widetilde{\delta A}_\theta^{\mathcal{R}} - \dot{\zeta}_0 \widetilde{\delta A}_\zeta^{\mathcal{R}}), \tag{72a}$$

$$G_1^{U,\mathcal{R}} \approx \delta\Phi^{\mathcal{R}}, \tag{72b}$$

$$G_1^{P,\mathcal{R}} \approx \delta A_\zeta^{\mathcal{R}}; \tag{72c}$$

$$G_1^{M,\mathcal{I}} = 0, \tag{73a}$$

$$G_1^{U,\mathcal{I}} = \langle \delta\Phi^{\mathcal{I}} \rangle, \tag{73b}$$

$$G_1^{P,\mathcal{I}} = \langle \delta A_\zeta^{\mathcal{I}} \rangle. \tag{73c}$$

In writing Eq. (73a), we have used Eq. (67) to obtain

$$\langle \delta \mathbf{A}^{\mathcal{I}} \cdot \partial_\xi \boldsymbol{\rho} \rangle = \langle \boldsymbol{\nabla} \delta\alpha \cdot \partial_\xi \boldsymbol{\rho} \rangle = 0. \tag{74}$$

Now, we can write down the gauge invariant representation of GKE,

$$\begin{aligned} f = {} & f_0(M,U,P) + \delta f + (\delta\Phi^{\mathcal{R}} + \langle \delta\Phi^{\mathcal{I}} \rangle) \frac{\partial f_0}{\partial U} \\ & + (\delta A_\zeta^{\mathcal{R}} + \langle \delta A_\zeta^{\mathcal{I}} \rangle) \frac{\partial f_0}{\partial P} \\ & + \Big[\langle \delta \mathbf{A}^{\mathcal{R}} \cdot \partial_\xi \boldsymbol{\rho} \rangle + \frac{1}{B} (\widetilde{\delta\Phi}^{\mathcal{R}} - \dot{\psi}_0 \widetilde{\delta A}_\psi^{\mathcal{R}} - \dot{\theta}_0 \widetilde{\delta A}_\theta^{\mathcal{R}} \\ & - \dot{\zeta}_0 \widetilde{\delta A}_\zeta^{\mathcal{R}}) \Big] \frac{\partial f_0}{\partial M}, \end{aligned} \tag{75a}$$

$$\frac{d}{dt} \delta f = -(\partial_U f_0 \partial_t - \partial_P f_0 \partial_\zeta)(\overline{H}_1^{\mathcal{R}} + \overline{H}_1^{\mathcal{I}}), \tag{75b}$$

$$\overline{H}_1^{\mathcal{R}} \equiv \langle \delta\Phi^{\mathcal{R}} \rangle - \dot{\psi}_0 \langle \delta A_\psi^{\mathcal{R}} \rangle - \dot{\theta}_0 \langle \delta A_\theta^{\mathcal{R}} \rangle - \dot{\zeta}_0 \langle \delta A_\zeta^{\mathcal{R}} \rangle - \dot{\xi}_0 \langle \delta \mathbf{A}^{\mathcal{R}} \cdot \partial_\xi \boldsymbol{\rho} \rangle, \tag{75c}$$

$$\overline{H}_1^{\mathcal{I}} \equiv \langle \delta\Phi^{\mathcal{I}} \rangle - \dot{\psi}_0 \langle \delta A_\psi^{\mathcal{I}} \rangle - \dot{\theta}_0 \langle \delta A_\theta^{\mathcal{I}} \rangle - \dot{\zeta}_0 \langle \delta A_\zeta^{\mathcal{I}} \rangle. \tag{75d}$$

In writing Eq. (75d), we have used Eq. (74).

Finally, we are at the position to verify that the perturbed distribution f in terms of guiding-center coordinates obtained by Eq. (75) is really invariant with respect to the gauge transformation. This is accomplished as follows.

Using Eqs. (63) and (64), we write Eq. (75d) as

$$\overline{H}_1^{\mathcal{I}} \equiv \langle \delta\Phi^{\mathcal{I}} \rangle - \dot{\psi}_0 \langle \delta A_\psi^{I} \rangle - \dot{\theta}_0 \langle \delta A_\theta^{\mathcal{I}} \rangle - \dot{\zeta}_0 \langle \delta A_\zeta^{\mathcal{I}} \rangle \tag{76a}$$

$$= -\frac{d}{dt} \langle \delta\alpha \rangle. \tag{76b}$$

Clearly, δf can be decomposed as

$$\delta f = \delta f^{\mathcal{R}} + \delta f^{\mathcal{I}}, \tag{77a}$$

$$\frac{d}{dt} \delta f^{\mathcal{R}} = -(\partial_U f_0 \partial_t - \partial_P f_0 \partial_\zeta) \overline{H}_1^{\mathcal{R}}, \tag{77b}$$

$$\delta f^{\mathcal{I}} = (\partial_U f_0 \partial_t - \partial_P f_0 \partial_\zeta) \langle \delta\alpha \rangle \tag{77c}$$

$$= -\langle \delta\Phi^{\mathcal{I}} \rangle \partial_U f_0 - \langle \delta A_\zeta^{\mathcal{I}} \rangle \partial_P f_0. \tag{77d}$$

In writing Eq. (77d), we have used Eqs. (63) and (64). Combining Eqs. (75a) and (77), we obtained

$$\begin{aligned} f = {} & f_0 + \delta f^{\mathcal{R}} + \delta\Phi^{\mathcal{R}} \frac{\partial f_0}{\partial U} + \delta A_\zeta^{\mathcal{R}} \frac{\partial f_0}{\partial P} + \Big[\langle \delta \mathbf{A}^{\mathcal{R}} \cdot \partial_\xi \boldsymbol{\rho} \rangle + \frac{1}{B} (\widetilde{\delta\Phi}^{\mathcal{R}} \\ & - \dot{\psi}_0 \widetilde{\delta A}_\psi^{\mathcal{R}} - \dot{\theta}_0 \widetilde{\delta A}_\theta^{\mathcal{R}} - \dot{\zeta}_0 \widetilde{\delta A}_\zeta^{\mathcal{R}}) \Big] \frac{\partial f_0}{\partial M}, \end{aligned} \tag{78a}$$

$$\frac{d}{dt} \delta f^{\mathcal{R}} = -(\partial_U f_0 \partial_t - \partial_P f_0 \partial_\zeta) \overline{H}_1^{\mathcal{R}}, \tag{78b}$$

$$\overline{H}_1^{\mathcal{R}} \equiv \langle \delta\Phi^{\mathcal{R}} \rangle - \dot{\psi}_0 \langle \delta A_\psi^{\mathcal{R}} \rangle - \dot{\theta}_0 \langle \delta A_\theta^{\mathcal{R}} \rangle - \dot{\zeta}_0 \langle \delta A_\zeta^{\mathcal{R}} \rangle - \dot{\xi}_0 \langle \delta \mathbf{A}^{\mathcal{R}} \cdot \partial_\xi \boldsymbol{\rho} \rangle. \tag{78c}$$

Therefore, we have proved that the final perturbed distribution function f in terms of guiding-center coordinates is independent of $(\delta\Phi^{\mathcal{I}}, \delta\mathbf{A}^{\mathcal{I}})$.

Noting that gauge transformation is equivalent to choosing different $(\delta\Phi^{\mathcal{I}}, \delta\mathbf{A}^{\mathcal{I}})$, one can immediately conclude that the new GKE, Eq. (75), is really gauge invariant.

A particular example may be useful for the readers to understand the importance of gauge invariance. Consider $(\delta\Phi, \delta\mathbf{A}) = (\delta\Phi^{\mathcal{I}}, \delta\mathbf{A}^{\mathcal{I}})$ or $(\delta\Phi^{\mathcal{R}}, \delta\mathbf{A}^{\mathcal{R}}) = (0,0)$. For this particular case, our gauge invariant GKE, Eq. (75), gives $f = f_0$, no perturbation to the distribution function. This can be easily understood as follows. Since $(\delta\Phi, \delta\mathbf{A}) = (\delta\Phi^{\mathcal{I}}, \delta\mathbf{A}^{\mathcal{I}})$ gives $(\delta\mathbf{E}, \delta\mathbf{B}) = (0,0)$, the system is physically at its equilibrium state. In this sense, $(\delta\Phi^{\mathcal{I}}, \delta\mathbf{A}^{\mathcal{I}})$ may be referred to as the ''imagined'' perturbation to the scalar and vector potentials; they do not have any contribution to the perturbation of electromagnetic fields, $(\delta\mathbf{E}, \delta\mathbf{B})$.

Now, it is clear that the previous approximate [4,7] S_1 [see, Eqs. (30) and (31)] is only proper for the Coulomb gauge $[\boldsymbol{\nabla} \cdot \delta\mathbf{A} = 0;\ (\delta\Phi^{\mathcal{I}}, \delta\mathbf{A}^{\mathcal{I}}) = (0,0)]$. The combination of Eqs. (37) and (32) is equivalent to ignoring the superscript $\mathcal{R}$ in Eq. (78). With the Coulomb gauge chosen, the new gauge invariant GKE, Eq. (75), recovers the combination of Eqs. (37) and (32) and consequently recovers the previous GKE's [1,8] after having picked back the $\mathbf{V}_d \cdot \delta\mathbf{A}_\perp$ term neglected in Refs. [1,8].

In applying the gauge invariant GKE with chosen $(\delta\Phi,\delta\mathbf{A})$, one may decompose $(\delta\Phi,\delta\mathbf{A})$ into $(\delta\Phi^{\mathcal{R}},\delta\mathbf{A}^{\mathcal{R}})$ and $(\delta\Phi^{\mathcal{I}},\delta\mathbf{A}^{\mathcal{I}})$ by using the following relations:

$$\delta\mathbf{A}(\mathbf{r},t)=\delta\mathbf{A}^{\mathcal{R}}(\mathbf{r},t)+\delta\mathbf{A}^{\mathcal{I}}(\mathbf{r},t), \tag{79a}$$

$$\delta\Phi(\mathbf{r},t)=\delta\Phi^{\mathcal{R}}(\mathbf{r},t)+\delta\Phi^{\mathcal{I}}(\mathbf{r},t); \tag{79b}$$

$$\nabla^2\delta\alpha(\mathbf{r},t)=\boldsymbol{\nabla}\cdot\delta\mathbf{A}(\mathbf{r},t); \tag{79c}$$

$$\delta\mathbf{A}^{\mathcal{I}}(\mathbf{r},t)=\boldsymbol{\nabla}\,\delta\alpha(\mathbf{r},t), \tag{79d}$$

$$\delta\Phi^{\mathcal{I}}(\mathbf{r},t)=-\partial_t\delta\alpha(\mathbf{r},t). \tag{79e}$$

If $\boldsymbol{\nabla}\cdot\delta\mathbf{A}(\mathbf{r},t)=0$, one may use Eq. (78) with the superscript $\mathcal{R}$ in Eq. (78) ignored.

According to the above comparisons and discussions, Eq. (78) (with the superscript $\mathcal{R}$ ignored) agrees well with Ref. [1]; the only difference lies in the $\mathbf{V}_d\cdot\delta\mathbf{A}_\perp$ term neglected in Ref. [1], and we have shown that this neglected term can be picked back within the frame of Ref. [1]. Since the GKE presented in Ref. [1] is so well known and so widely used, we shall discuss the validity of this GKE in detail.

First, as we have pointed out in the last section, the $\mathbf{V}_d\cdot\delta\mathbf{A}_\perp$ term should be picked back when $\delta\mathbf{A}_\perp\neq 0$. After having picked back the $\mathbf{V}_d\cdot\delta\mathbf{A}_\perp$ term, the GKE in Ref. [1] is equivalent to Eq. (78) (with the superscript $\mathcal{R}$ ignored). It is useful to note that even if $\delta\mathbf{A}_\perp\neq 0$, neglecting the $\mathbf{V}_d\cdot\delta\mathbf{A}_\perp$ term still can be taken as a good approximation, provided that an appropriate gauge is chosen so that $\mathbf{V}_d\cdot\delta\mathbf{A}_\perp \ll V_\parallel\delta A_\parallel$. In other words, in neglecting the $\mathbf{V}_d\cdot\delta\mathbf{A}_\perp$ term, one has to add a constraint $\mathbf{V}_d\cdot\delta\mathbf{A}_\perp\ll V_\parallel\delta A_\parallel$ on the choice of gauge. Note that the gauge chosen in Eq. (56) does not satisfy the condition $\mathbf{V}_d\cdot\delta\mathbf{A}_\perp\ll V_\parallel\delta A_\parallel$.

Strictly, Eq. (78) (with the superscript $\mathcal{R}$ ignored) is correct only for the Coulomb gauge, $\boldsymbol{\nabla}\cdot\delta\mathbf{A}(\mathbf{r},t)=0$. In practice, Eq. (78) (with the superscript $\mathcal{R}$ ignored) may be taken as a good approximation, when the following condition is satisfied:

$$\boldsymbol{\nabla}\cdot\delta\mathbf{A}(\mathbf{r},t)\ll|\boldsymbol{\nabla}\times\delta\mathbf{A}(\mathbf{r},t)|. \tag{80}$$

Equation (80) may be referred to as an approximated Coulomb gauge. When $\boldsymbol{\nabla}\cdot\delta\mathbf{A}(\mathbf{r},t)\sim|\boldsymbol{\nabla}\times\delta\mathbf{A}(\mathbf{r},t)|$, Eq. (78) (with the superscript $\mathcal{R}$ ignored) is still correct for some particular problems. As we have pointed out in the beginning of this section, Eq. (78) (with the superscript $\mathcal{R}$ ignored) is not gauge invariant just because the gyrophase-dependent part of the perturbed distribution function is not gauge invariant. In other words, even if $\boldsymbol{\nabla}\cdot\delta\mathbf{A}(\mathbf{r},t)\neq 0$, Eq. (78) (with the superscript $\mathcal{R}$ ignored) still correctly gives the gyrophase-independent part of the perturbed distribution function, and consequently it still correctly gives the perturbed mass density, the perturbed charge density, and the perturbed parallel current density. In a word, it is always correct for shear Alfven wave problems.

We point out that it is easy to recover the gauge invariance in the previous modern GKE's by modifying the solution of S_1 (the gauge function for the Hamiltonian Lie transform) in the way similar to ours. For classical GKE's, noting that it is not clear what is equivalent in classical GKE to the S_1 function in modern GKE, and facing the dauntingly lengthy tensor analysis employed in Ref. [1], which is known as the most concise presentation of the derivation of classical GKE's, we have not tried to consider how to recover the gauge invariance in the classical GKE's. This may be left as a topic for future investigation.

To close this section, we briefly summarize the three limiting cases of the gauge invariant representation of GKE, Eq. (75).

The eikonal form of the gauge invariant GKE reads

$$\begin{aligned} f={}&f_0(M,U,P)+\delta f+(\delta\Phi^{\mathcal{R}}+J_0\delta\Phi_c^{\mathcal{I}})\frac{\partial f_0}{\partial U}\\ &+(\delta A_\zeta^{\mathcal{R}}+J_0\delta A_{\zeta c}^{\mathcal{I}})\frac{\partial f_0}{\partial P}+\frac{1}{B}\bigg[-\frac{J_1(k_\perp\rho)}{k_\perp\rho/2}M\,\delta B_c\\ &+(\delta\Phi^{\mathcal{R}}-J_0\delta\Phi_c^{\mathcal{R}})-\frac{d\mathbf{R}}{dt}\cdot(\delta\mathbf{A}^{\mathcal{R}}-J_0\delta\mathbf{A}_c^{\mathcal{R}})\bigg]\frac{\partial f_0}{\partial M}, \end{aligned} \tag{81a}$$

$$\begin{aligned} \frac{d}{dt}\delta f={}&-(\partial_U f_0\partial_t-\partial_P f_0\partial_\zeta)\bigg[J_0\bigg(\delta\Phi_c-\frac{d\mathbf{R}}{dt}\cdot\delta\mathbf{A}_c\bigg)\\ &+\frac{J_1(k_\perp\rho)}{k_\perp\rho/2}M\,\delta B_c\bigg]. \end{aligned} \tag{81b}$$

The DKE limit of the gauge invariant GKE is the same as Eq. (46).

The small-banana-width limit of the gauge invariant GKE is

$$f=f_0(M,U,\psi)+\delta f+G_1^M\partial_M f_0+G_1^U\partial_U f_0-G_1^P\partial_\psi f_0, \tag{82a}$$

$$\frac{d}{dt}\delta f=-(\partial_U f_0\partial_t+\partial_\psi f_0\partial_\zeta)(\overline{H_1^{\mathcal{R}}}+\overline{H_1^{\mathcal{I}}}) \tag{82b}$$

with (G_1^M,G_1^U,G_1^P) given by Eq. (65c) and Eqs. (72) and (73) and $(\overline{H_1^{\mathcal{R}}},\overline{H_1^{\mathcal{I}}})$ given by Eqs. (75c,d).

VIII. SUMMARY AND DISCUSSIONS

We have systematically identified the canonical gyrocenter variables by using the canonical Hamiltonian Lie-transform perturbation method. With the canonical gyrocenter variables, we have established a new representation of GKE in terms of the magnetic moment M, the total energy U, and the canonical toroidal momentum P. The new representation is invariant with respect to the gauge transformation of perturbation fields. In the new representation of GKE, the effects of toroidal symmetry breaking are explicitly revealed. Every term involved in the GKE presented in the new representation has its own unambiguous physical interpretation. Transformed back to the guiding-center coordinates, the adiabatic part of the perturbed distribution function is completely determined by the perturbations of (M,U,P). The nonadiabatic part of the distribution function δf is completely determined by the time rates of change of U and P due to the time dependence and toroidal angle dependence introduced by the perturbation of fields. It is clearly and rigorously shown that the free energy associated with $\partial_P f_0(M,U,P)$ does not have any nonadiabatic linear driving to any axisymmetric modes, which is an important issue in axisymmetric tokamak plasma physics.

There are two papers devoted to represent the GKE[10] or DKE[11] in a similar way. In Ref. [10], the eikonal ansatz was used, so that it cannot be used for electromagnetic modes with arbitrary mode numbers. Moreover, it is not clear in Ref. [10] what kind of roles the perturbed canonical toroidal momentum plays in the gyrokinetic theory, and the physical interpretation of the adiabatic part of perturbed distribution function defined in Ref. [10] is not as simple and clear as in the new representation developed in this paper. In addition, the GKE in Ref. [10] is correct only for a gauge chosen so that $\mathbf{V}_d\cdot\delta\mathbf{A}_\perp=0$ and $\nabla\cdot\delta\mathbf{A}=0$, or approximately correct when an appropriate gauge is chosen so that $\mathbf{V}_d\cdot\delta\mathbf{A}_\perp\ll V_\parallel\delta A_\parallel$ and $\nabla\cdot\delta\mathbf{A}\ll|\nabla\times\delta\mathbf{A}|$. In Ref. [11], the FLR effects were dropped in obtaining the DKE. Therefore, the previous theories [10,11] shall be regarded as limiting cases of the new representation of GKE. And indeed, we have verified that in the corresponding limits, the new representation of the GKE recovers the previous results [10,11]. We have also provided the small-banana-width version of the new representation of GKE.

The $\mathbf{V}_d\cdot\delta\mathbf{A}_\perp$ term neglected in previous theories [1–10] has been systematically retained in this representation of GKE. It has been shown that retaining the $\mathbf{V}_d\cdot\delta\mathbf{A}_\perp$ term is important to make this representation of GKE invariant with respect to the gauge transformation of the perturbation fields, in contrast to the previous theories [1–10], whose gauge variance has not been well recognized and discussed so far. And we have shown how to pick back the $\mathbf{V}_d\cdot\delta\mathbf{A}_\perp$ term in the previous theories [1,8]. We also note that the $\mathbf{V}_d\cdot\delta\mathbf{A}_\perp$ term can be dropped out of the GKE with an appropriate gauge chosen to satisfy $\mathbf{V}_d\cdot\delta\mathbf{A}_\perp\ll V_\parallel\delta A_\parallel$.

As we have shown, even for ideal MHD shear Alfven wave perturbations, gauge invariance can provide considerable convenience. For high beta (ratio of plasma pressure to magnetic field pressure) tokamaks, effects of compressional Alfven wave may be important. In this case, usually we have both finite $\delta\mathbf{A}_\parallel$ and finite $\delta\mathbf{A}_\perp$. If one recognizes that the previous GKE's [1–10] have to be used with the Coulomb gauge chosen (note that it has not been well recognized so far; among Refs. [1–10], only Ref. [2] pointed out that the Coulomb gauge should be chosen in applying the GKE presented there) and still insists to use the previous GKE's, first, one has to carefully choose the Coulomb gauge ($\nabla\cdot\delta\mathbf{A}=0$) or at least an approximate Coulomb gauge ($\nabla\cdot\delta\mathbf{A}\ll|\nabla\times\delta\mathbf{A}|$) and second, one has to pick back the $\mathbf{V}_d\cdot\delta\mathbf{A}_\perp$ term in the way discussed in the main text of this paper or make sure that the chosen gauge satisfies $\mathbf{V}_d\cdot\delta\mathbf{A}_\perp\ll V_\parallel\delta A_\parallel$. But in general, it may be inconvenient to use the approximate Coulomb gauge satisfying $\mathbf{V}_d\cdot\delta\mathbf{A}_\perp\ll V_\parallel\delta A_\parallel$. Clearly, for this case, the gauge invariant representation of GKE shall provide considerable convenience.

For readers who may wish to reexamine those works based on the previous gauge variant GKE's, we propose the following criteria.

(1) $\delta\mathbf{A}=0$ (electrostatic modes).

(2) $\nabla\cdot\delta\mathbf{A}=0$ or $\nabla\cdot\delta\mathbf{A}\ll|\nabla\times\delta\mathbf{A}|$, and $\mathbf{V}_d\cdot\delta\mathbf{A}_\perp=0$ or $\mathbf{V}_d\cdot\delta\mathbf{A}_\perp\ll V_\parallel\delta A_\parallel$.

(3) $\mathbf{V}_d\cdot\delta\mathbf{A}_\perp=0$ or $\mathbf{V}_d\cdot\delta\mathbf{A}_\perp\ll V_\parallel\delta A_\parallel$, and $\nabla\cdot\delta\mathbf{A}\sim|\nabla\times\delta\mathbf{A}|$, but the given physical problem does not involve the odd moments of $\mathbf{v}_\perp$ (for example, the perturbed perpendicular current density).

(4) $\omega_*\propto\partial_\zeta$ is used.

(5) $\omega_*\propto\partial_\theta$ is used, but $\omega_*\propto\partial_\theta$ is equivalent to $\omega_*\propto\partial_\zeta$.

If all of (1)–(3) are not satisfied or (4) and (5) are not satisfied, then the work based on the previous gauge variant GKE's may need careful reconsideration. Of course, we expect that most of those works based on the previous gauge variant GKE's satisfy one of (1)–(3) and one of (4) and (5), and consequently do not need any reconsideration from our viewpoint.

ACKNOWLEDGMENTS

The author is indebted to Dr. H. L. Berk and Dr. H. V. Wong, who carefully read the manuscript and suggested the author investigate the issue of gauge invariance and directly verify the gauge invariance of the final gyrokinetic equation. The author would like to thank Dr. C. Z. Cheng for stimulating discussions. Thanks are also given to the National Natural Sciences Foundation of China for financial support under Grant No. 19705008.

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